The Fission, Fusion and Annihilation of Solitons of the (2+1)-Dimensional Broer-Kaup-Kupershmidt System

Song-Hua Ma, Jian-Ping Fang, and Chun-Long Zheng

Department of Physics, Zhejiang Lishui University, Lishui 323000, China

Reprint requests to J.-P. F.; E-mail: zjlsfjp@yahoo.com.cn

Z. Naturforsch. **62a,** 8 – 12 (2007); received November 10, 2006

By means of an improved mapping approach, a series of excitations of the (2+1)-dimensional Broer-Kaup-Kupershmidt (BKK) system is derived. Based on the derived solitary wave excitation, some specific fission, fusion and annihilation phenomena of solitons are also obtained. — PACS numbers: 05.45.Yv, 03.65.Ge.

Key words: Improved Mapping Approach; BKK System; Fission; Fusion; Annihilation.

1. Introduction

The interactions between soliton solutions of integrable models are usually considered to be completely elastic. That is to say, the amplitude, velocity and wave shape of solitons are not changed after nonlinear interaction [1]. However, for some specific solutions of some (2+1)-dimensional models, the interactions among solitonic excitations are not completely elastic since their shapes are changed after their collisions [2]. Furthermore, for some nonlinear models, two or more solitons may fuse into one soliton at a specific time, while sometimes one soliton may fission into two or more solitons at another specific time [3]. These phenomena are often called soliton fusion and soliton fission, respectively, and have been observed in many kinds of physical systems such as some organic membranes and macromolecular materials [4], and in many physical fields like plasma physics, nuclear physics and hydrodynamics [5]. Recently, Wang *et al.* [6] first discussed two (1+1)-dimensional models, the Burgers equation and Sharma-Tasso-Olver (STO) equation, via Hirota's direct method and the Bäcklund transformation, and found the soliton fission and soliton fusion phenomenas. Furthermore, Ma and Zheng [7] also fond these phenomena in the (2+1)-dimensional Higher-Order-Broer-Kaup (HBK) system. Along this line, in this paper, we further study the soliton fission, soliton fusion phenomena and the annihilation of solitons in the following celebrated (2+1)-dimensional Broer-Kaup-Kupershmidt (BKK) system [8]:

$$u_{ty} - u_{xxy} + 2(u_x u)_y + 2v_{xx} = 0,$$

$$v_t + 2(uv)_x + v_{xx} = 0. \quad (1)$$

The BKK system is used to model the nonlinear and dispersive long gravity waves travelling in two horizontal directions in shallow water with uniform depth, and it can also be derived from the celebrated Kadomtsev-Petviashvili (KP) equation by the symmetry constraint [9]. When $y = x$, the (2+1)-dimensional BKK system is reduced further to a usual (1+1)-dimensional BKK system, which can be used to describe the propagation of long waves in shallow water [10]. Using some suitable dependent and independent variable transformations, Chen and Li [11] have proved that the (2+1)-dimensional BKK system can be transformed to the (2+1)-dimensional dispersive long wave equation (DLWE) and (2+1)-dimensional Ablowitz-Kaup-Newell-Segur (AKNS) system. The (2+1)-dimensional BKK system has been widely investigated in detail by many researchers [12].

2. New Exact Solutions of the (2+1)-Dimensional BKK System

As is well known, to search for the solitary wave solutions of a nonlinear physical model, we can apply different approaches. One of the most efficient methods of finding soliton excitations of a physical model is the so-called improved mapping approach. The basic ideal of the algorithm is as follows. For a given nonlinear partial differential equation (NPDE) with the independent variables x $(= x_0 = t, x_1, x_2, \cdots, x_m)$, and the dependent variable u, in the form

$$P(u, u_t, u_{x_i}, u_{x_i x_j}, \cdots) = 0, \quad (2)$$

0932–0784 / 07 / 0100–0008 $ 06.00

where P is in general a polynomial function of its arguments, and the subscripts denote the partial derivatives, the solution can be assumed to be in the form

$$u = A(x) + \sum_{i=1}^{n} B_i(x)\phi^i[q(x)] + \frac{C_i}{\phi^i[q(x)]} + D_i(x)\phi^{i-1}[q(x)]\sqrt{\sigma + \phi^2[q(x)]} + \frac{E_i}{\phi^{i-1}[q(x)]\sqrt{\sigma + \phi^2[q(x)]}} \quad (3)$$

with

$$\phi' = \sigma + \phi^2, \quad (4)$$

where σ is a constant and the prime denotes the differentiation with respect to q. To determine u explicitly, one may substitute (3) and (4) into the given NPDE and collect coefficients of polynomials of ϕ, then eliminate each coefficient to derive a set of partial differential equations of A, B_i, C_i, D_i, E_i, and q, and solve the system of partial differential equations to obtain A, B_i, C_i, D_i, E_i, and q. Finally, (4) possesses the general solution (the tanh-type and tan-type solutions are neglected here)

$$\begin{aligned} &\text{(a)} \quad \phi = -\sqrt{-\sigma}\coth(\sqrt{-\sigma}q), && \text{when } \sigma < 0,\\ &\text{(b)} \quad \phi = -\sqrt{\sigma}\cot(\sqrt{\sigma}q), && \text{when } \sigma > 0,\\ &\text{(c)} \quad \phi = \tfrac{-1}{q}, && \text{when } \sigma = 0. \end{aligned} \quad (5)$$

Substituting A, B_i, C_i, D_i, E_i, q and (5) into (3), one can obtain the exact solutions of the given NPDE.

First, let us make a transformation: $v = u_y$. Substituting this transformation in (1) yields

$$u_{yt} + 2(u_x u)_y + u_{xxy} = 0. \quad (6)$$

Now we apply the improved mapping approach to (6). Similarly to the usual mapping approach [13], we can determine $n = 1$ by balancing the highest-order nonlinear term with the highest-order partial derivative term in (6), and ansatz (3) becomes

$$u = f + g\phi + \frac{h}{\phi} + G\sqrt{\sigma + \phi^2} + \frac{H}{\sqrt{\sigma + \phi^2}}, \quad (7)$$

where f, g, h, G, H, and q are functions of (x,y,t) to be determined. Substituting (7) and (4) in (6) and collecting the coefficients of the polynomials of ϕ, then setting each coefficient to zero, we have

$$f = -\frac{1}{2}\frac{q_t + q_{xx}}{q_x}, \quad g = -\frac{1}{2}q_x, \quad h = -\frac{1}{2}q_y, \quad G = \frac{1}{2}q_x, \quad H = \frac{1}{2}q_y \quad (8)$$

with

$$q = \chi(x,t) + \varphi(y), \quad (9)$$

where $\chi \equiv \chi(x,t)$, $\varphi \equiv \varphi(y)$ are two arbitrary variable separation functions of (x,t) and of y, respectively. Based on the solutions of (4), one thus obtains an explicit solution of (1).

Case 1. For $\sigma = -1$, we can derive the following solitary wave solutions of (1):

$$u_1 = -\frac{1}{2}\frac{\chi_{xx} + \chi_t - \chi_x^2\coth(\chi + \varphi)}{\chi_x} + \frac{1}{2}\chi_x\operatorname{csch}(\chi + \varphi), \quad (10)$$

$$v_1 = \frac{1}{2}\chi_x\varphi_y\big(-\operatorname{csch}(\chi + \varphi)^2 - \operatorname{csch}(\chi + \varphi)\coth(\chi + \varphi)\big). \quad (11)$$

Case 2. For $\sigma = 1$, we obtain the following periodic wave solutions of (1):

$$u_2 = -\frac{1}{2}\frac{\chi_{xx} + \chi_t - \chi_x^2\cot(\chi + \varphi)}{\chi_x} + \frac{1}{2}\chi_x\csc(\chi + \varphi), \quad (12)$$

$$v_2 = -\frac{1}{2}\chi_x\varphi_y\big(\csc(\chi + \varphi)^2 + \csc(\chi + \varphi)\cot(\chi + \varphi)\big). \quad (13)$$

Case 3. For $\sigma = 0$, we find the following variable separated solution of (1):

$$u_3 = -\frac{1}{2}\frac{\chi_t + \chi_{xx}}{\chi_x} + \frac{\chi_x}{\chi + \varphi}, \quad (14)$$

$$v_3 = -\frac{\varphi_y\chi_x}{(\chi + \varphi)^2}. \quad (15)$$

3. The Fission, Fusion and Annihilation of Solitons

In this section, we mainly discuss the solitary solutions, namely Case 1. Owing to the arbitrariness of

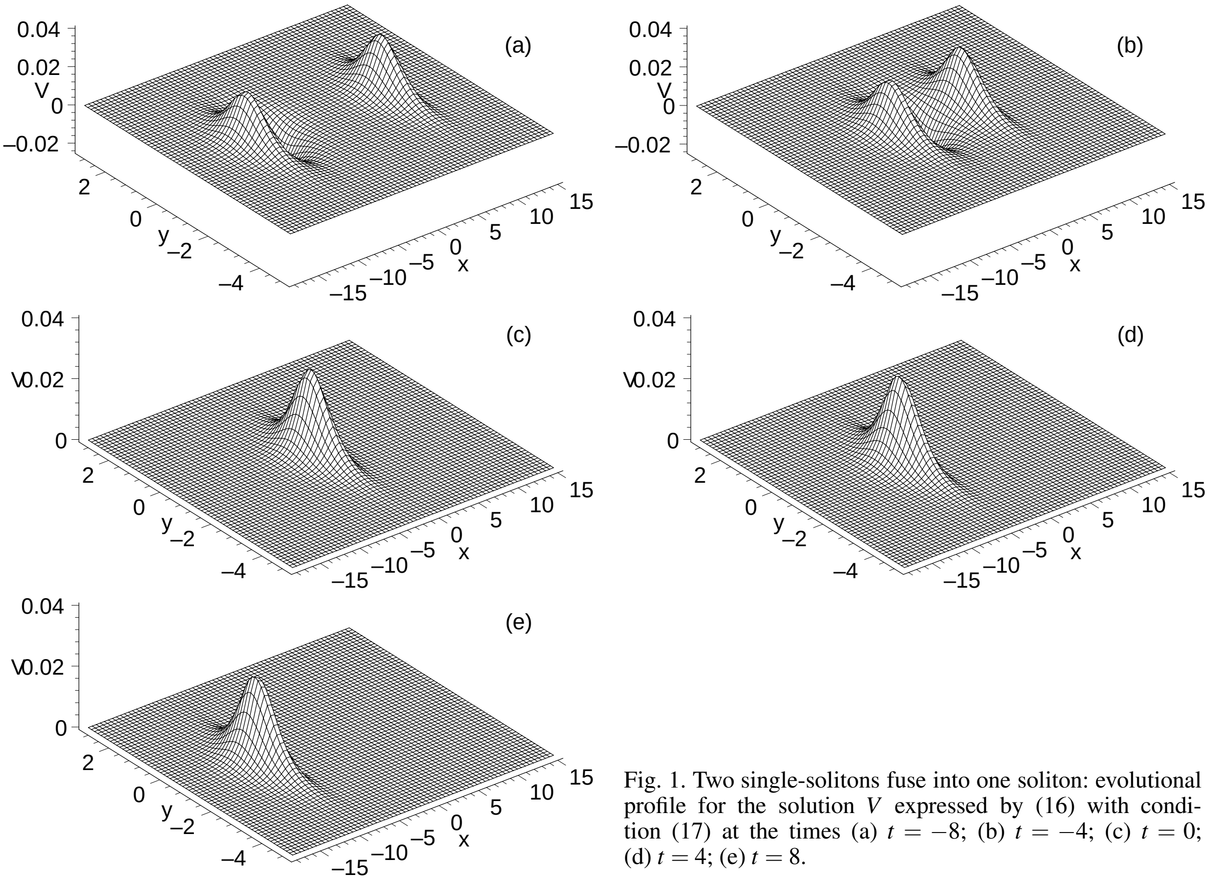

Fig. 1. Two single-solitons fuse into one soliton: evolutional profile for the solution V expressed by (16) with condition (17) at the times (a) $t=-8$; (b) $t=-4$; (c) $t=0$; (d) $t=4$; (e) $t=8$.

the functions $\chi(x,t)$ and $\varphi(y)$ included in this case, the physical quantities u and v may possess rich localized structures. For simplicity in the following discussion, we merely analyze the solitary wave excitation v_1 expressed by (11) and rewrite it in a simple form, namely

$$V=v_1=\frac{1}{2}\chi_x\varphi_y\big(-\operatorname{csch}(\chi+\varphi)^2-\operatorname{csch}(\chi+\varphi)\cdot\coth(\chi+\varphi)\big). \tag{16}$$

Now we focus our attention on these intriguing fusion, fission and annihilation phenomena for the solitary wave solutions V, which may exist under certain circumstances. For instance, when we select the arbitrary functions χ and φ to be

$$\chi=1+\exp(x+t)+\operatorname{sech}(x-t),\quad \varphi=1+\tanh(y), \tag{17}$$

and substitute (17) in (16), we can obtain a new kind of solitary solution of (1). Figure 1 shows an evolutional profile corresponding to the physical quantity V of the two-dromion solution expressed by (16), exhibiting a fusion phenomenon for the two solitons. From Fig. 1, we can clearly see that the two single-solitons fuse to one soliton finally.

Along with this line, as we consider χ and φ to be

$$\chi=1+\operatorname{sech}(x+t)+\exp(x-t),\quad \varphi=1+\operatorname{sech}(y), \tag{18}$$

we can obtain another new type of solitary excitation with apparently different properties presented in Fig. 2, comparing with the fusion phenomenon in Figure 1. From Fig. 2, we can find that one single-soliton fissions into two solitons.

Just like other particles, solitons can also be annihilated under some appropriate conditions. For example, when choosing $\chi(x,t)$ and $\varphi(y)$ in solution (16) to be

$$\chi=1+\operatorname{sech}(x^2+t),\quad \varphi=1+\operatorname{sech}(y^2), \tag{19}$$

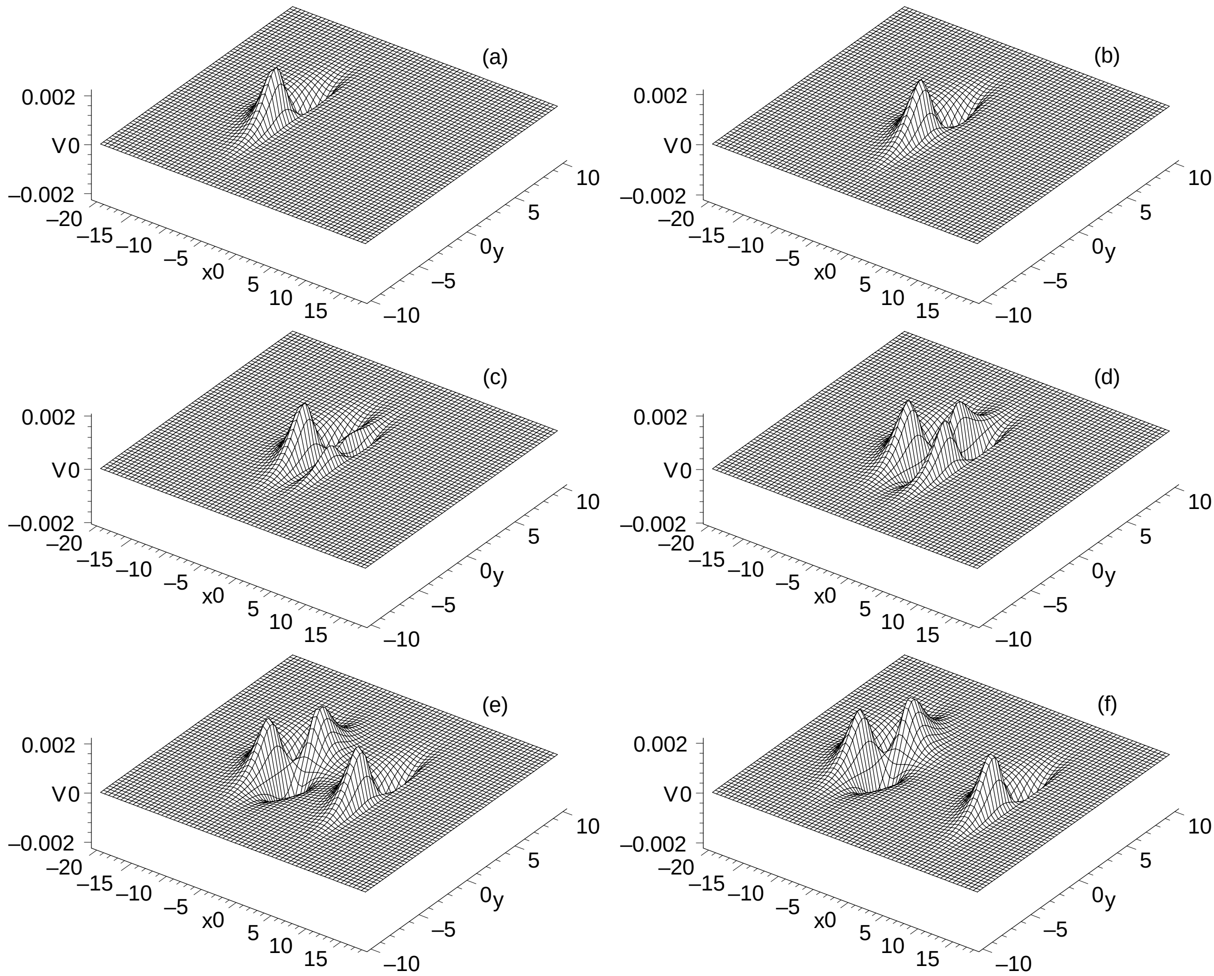

Fig. 2. One single-soliton fissions into two solitons; evolutional plot of the solution V expressed by (16) with condition (18) at the times (a) $t = -6$; (b) $t = 0$; (c) $t = 1$; (d) $t = 2$; (e) $t = 6$; (f) $t = 9$.

we can see the annihilation of solitons for the physical quantity V of (16) under the condition (19) presented in Fig. 3 with fixed parameters $t = -7, 0, 2, 3, 5, 7$. From Fig. 3, we find that the amplitude and shape of the solitons become smaller and smaller after interactions, finally, they reduce to zero.

4. Summary and Discussion

In this paper, via the improved mapping approach, we have found new exact solutions of the BKK system. Based on the derived solitary wave excitation, we have studied the fission, fusion and annihilation phenomena of solitons. And we can see that solitons have the same characteristics like other particles in many aspects.

Although we have given out some soliton fusion, fission, and annihilation phenomena in the (2+1)-dimensional case, it is obvious that there are still many significant and interesting problems to be further discussed. As the author [14] have pointed out in (1+1)-dimensional cases: What is the necessary and sufficient condition for soliton fusion and fission? What is for soliton elastic and nonelastic interaction? What is the general equation for the distribution of the energy and momentum after soliton fusion and soliton fission? How can we use the soliton fusion and soliton fission of integrable models to investigate the observed soliton fusion and soliton fission in the experiments? These are all the pending issues to be further studied.

Acknowledgement

The authors would like to thank the anonymous referee for his helpful suggestions and positive comments.

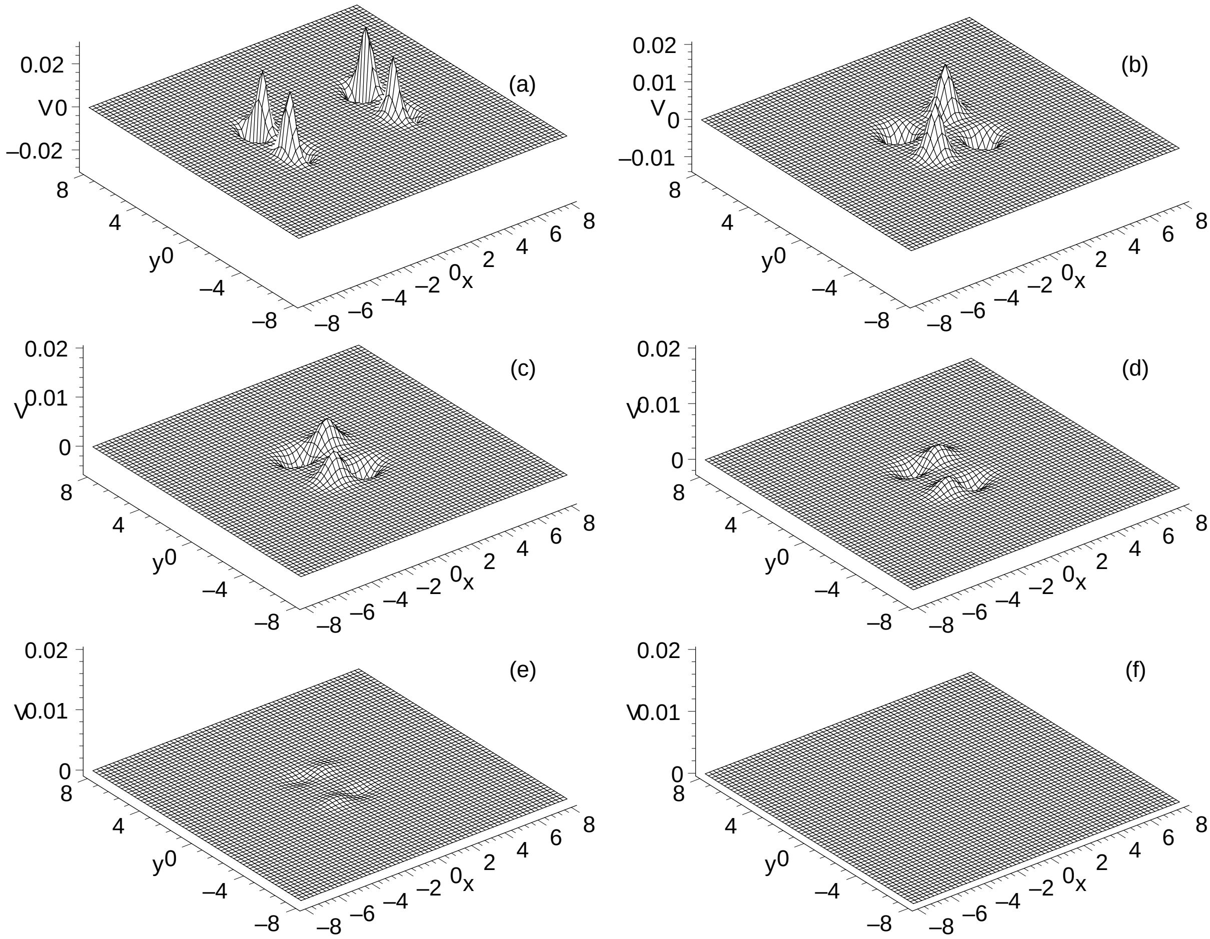

Fig. 3. Plot of the annihilation of solitons for the solution V expressed by (16) under the condition (19) at the times (a) $t = -7$; (b) $t = 0$; (c) $t = 2$; (d) $t = 3$; (e) $t = 5$; (f) $t = 7$.

The project was supported by the Natural Science Foundation of Zhejiang Province (Grant No. Y604106) and the Natural Science Foundation of Zhejiang Lishui University under Grant No. KZ05010.

[1] X. Y. Tang and S. Y. Lou, Phys. Rev. E **66**, 046601 (2002); X. Y. Tang and S. Y. Lou, J. Math. Phys. **44**, 4000 (2003).
[2] S. Y. Lou, J. Phys. A. Math. Gen. **35**, 10619 (2002); C. L. Zheng, Chin. Phys. **12**, 472 (2003).
[3] J. P. Ying, Commun. Theor. Phys. **35**, 405 (2001).
[4] V. N. Serkin, Opt. Commun. **192**, 237 (2001).
[5] G. Stoitchena, Math. Comput. Simul. **55**, 621 (2001).
[6] S. Wang, Soliton fission and fusion: Burgers equation and Sharma-Tasso-Olver equation, Bachelor Thesis, Ningbo University 2001; S. Wang, X. Y. Tang, and S. Y. Lou, Chaos Solitons and Fractals **19**, 769 (2004).
[7] Z. Y. Ma and C. L. Zheng, Commun. Theor. Phys. **43**, 994 (2005).
[8] V. G. Durovsky and E. G. Konopelchenko, J. Phys. A **27**, 4619 (1994); M. Boiti, Inv. Prob. **3**, 37 (1987).
[9] S. Y. Lou and X. B. Hu, J. Math. Phys. **38**, 6401 (1997).
[10] V. E. Zakharov and L. Li, Appl. Mech. Tech. Phys. **9**, 190 (1998).
[11] C. L. Chen and Y. S. Li, Theor. Phys. **38**, 129 (2002).
[12] J. P. Ying and S. Y. Lou, Z. Naturforsch. **56a**, 619 (2001); J. F. Zhang and P. Han, Acta Phys. Sin. **21**, 705 (2002); H. M. Li, Commun. Theor. Phys. **39**, 513 (2003).
[13] J. P. Fang and C. L. Zheng, Z. Naturforsch. **60a**, 245 (2005).
[14] S. Wang, Y. S. Xu, and S. Y. Lou, Chin. Phys. **14**, 1049 (2003).